Closing Tues: 12.1,12.2,12.3
Closing Thurs: 12.4(1),12.4(2),12.5(1)

Vector operations so far

Scalar multiplication:

c **v** = "vector parallel to **v** with length scaled by a factor of *c*"

Vector Addition:

**a** + **b** = **"**if **a** and **b** are drawn tail to head, then **a** + **b** is the vector that goes from the tail of **a** to the head of **b**"
(resultant/combined force)

## 12.3 Dot Products

If **a =** < $a_1$, $a_2$, $a_3$ > and **b** = <$b_1$, $b_2$, $b_3$>
Then we define the dot product by:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

*Note*: The dot product gives a number (scalar).

*Entry Task*: **a** = < 3, 1, 2 >, c = 4
**b** = - **i** + 6**j** + 5**k**

Compute

1. c**a**
2. unit vector in the direction of **a.**
3. **a** + **b**
4. **a** · **b**

**Basic fact list**:

- Manipulation facts
  (works like regular multiplication):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$\mathrm{c}(\mathbf{a} \cdot \mathbf{b}) = (\mathrm{c}\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathrm{c}\mathbf{b})$$

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = ???$$

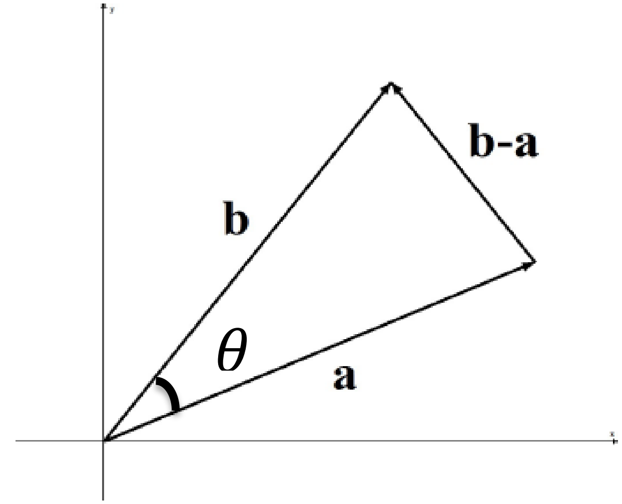
- Helpful fact:

$$\mathbf{a} \cdot \mathbf{a} = \mathrm{a}_1^2 + a_2^2 + a_3^2 = |\boldsymbol{a}|^2$$

**The most important fact**:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta),$$

where $\theta$ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \leq \theta \leq \pi)$




*Proof* (not required):

By the Law of Cosines:

$$|\mathbf{b} - \mathbf{a}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)$$

The left-hand side expands to

$$\begin{aligned} |\mathbf{b} - \mathbf{a}|^2 &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^{\mathbf{2}} \end{aligned}$$

Subtracting $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$ from both sides gives

$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos(\theta).$$

Divide by -2 to get the result. (QED)

**Most important consequence**:

If **a** and **b** are orthogonal, then

$$\mathbf{a} \cdot \mathbf{b} = 0.$$

Also:

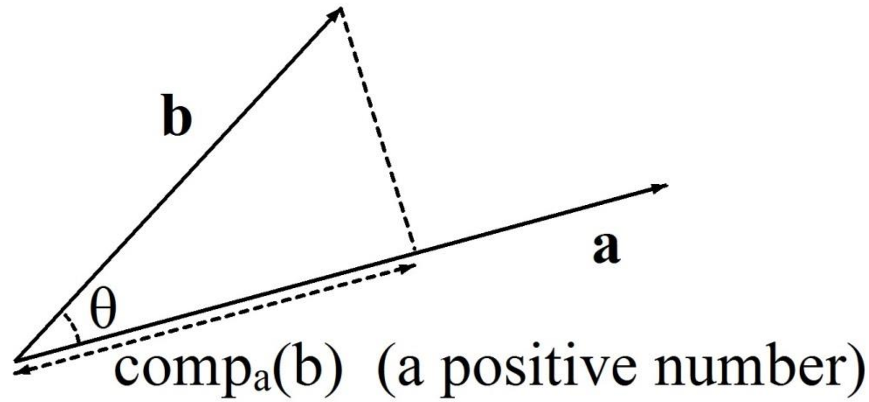
If **a** and **b** are parallel, then

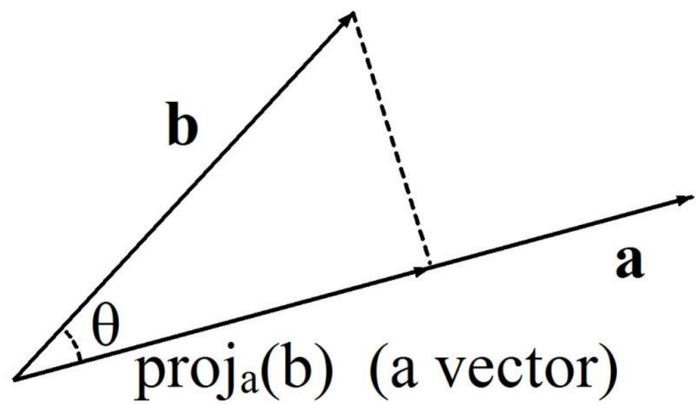
$$\mathbf{a} \cdot \mathbf{b} = |\mathrm{a}||\mathrm{b}|$$

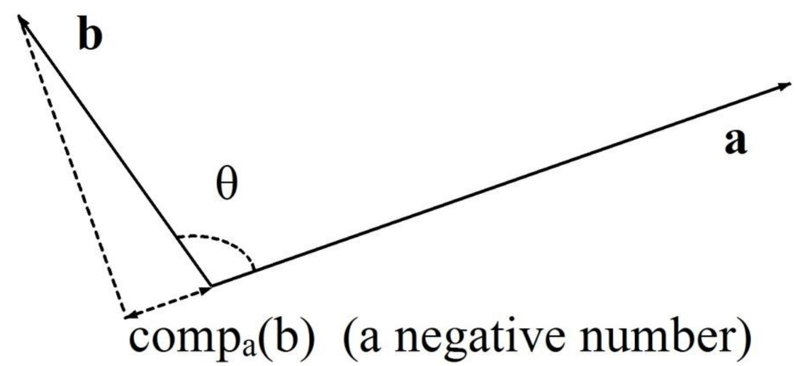
or

$$\mathbf{a} \cdot \mathbf{b} = -|\mathrm{a}||\mathrm{b}|.$$

*Projections*:



## 12.4 The Cross Product

We define the cross product, or vector product, for two 3-dimensional vectors, $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$, by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \boldsymbol{k}$$

$$= (a_2 b_3 - a_3 b_2)\boldsymbol{i} - (a_1 b_3 - a_3 b_1)\boldsymbol{j} + (a_1 b_2 - a_2 b_1)\boldsymbol{k}$$

*Ex*: If $\boldsymbol{a} = \langle 1,2,0 \rangle$ and $\boldsymbol{b} = \langle -1,3,2 \rangle$, then

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix} =$$

$$(\quad - \quad)\boldsymbol{i} - (\quad - \quad)\boldsymbol{j} + (\quad - \quad)\boldsymbol{k}$$

*You do* $\boldsymbol{a} = \langle 1,3,-1\rangle$, $\boldsymbol{b} = \langle 2,1,5\rangle$.
Compute $\boldsymbol{a} \times \boldsymbol{b}$

**Most important fact**:
The vector $\boldsymbol{v} = \mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**.

*Right-hand rule*
If the fingers of the right-hand curl from **a** to **b**, then the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$**.**

*The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$:*

Through some algebra and using the dot product rule, it can be shown that

$$|\mathbf{a} \times \mathbf{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin(\theta)$$

where $\theta$ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \leq \theta \leq \pi)$